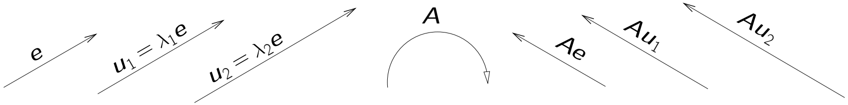
Applicazioni lineari (cenni)

- $\boldsymbol{A} : \mathcal{V} \to \mathcal{V}$, $\boldsymbol{A} : \boldsymbol{u} \mapsto \boldsymbol{v} := \boldsymbol{A}(\boldsymbol{u})$

- $\boldsymbol{A}$ lineare se per ogni $\lambda_1, \lambda_2 \in \mathbb{R}$ e per ogni $\boldsymbol{u}_1, \boldsymbol{u}_2 \in \mathcal{V}$, si ha $\boldsymbol{A}(\lambda_1 \boldsymbol{u}_1 + \lambda_2 \boldsymbol{u}_2) = \lambda_1 \boldsymbol{A}(\boldsymbol{u}_1) + \lambda_2 \boldsymbol{A}(\boldsymbol{u}_2)$

- indichiamo con $\boldsymbol{Au} = \boldsymbol{A}(\boldsymbol{u})$, e con $Lin(\mathcal{V})$ l'insieme delle applicazioni lineari di $\mathcal{V}$ in $\mathcal{V}$ (tensori del secondo ordine)

- Oss. $\boldsymbol{A}(\lambda \boldsymbol{e}) = \lambda \boldsymbol{A}(\boldsymbol{e})$ (vettori paralleli si trasformano in vettori paralleli)

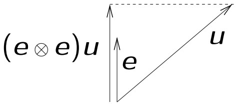


Applicazioni lineari (cenni)

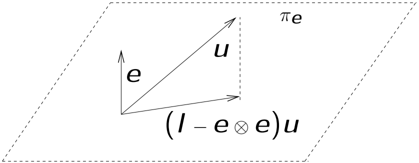
- applicazione identità: $\boldsymbol{I}\boldsymbol{u} = \boldsymbol{u}$ per ogni $\boldsymbol{u} \in \mathcal{V}$
- applicazione nulla: $\boldsymbol{O}\boldsymbol{u} = \boldsymbol{o}$ per ogni $\boldsymbol{u} \in \mathcal{V}$
- Somma: $\boldsymbol{A}, \boldsymbol{B} \in Lin \mapsto \boldsymbol{A} + \boldsymbol{B} \in Lin$, t.c. $(\boldsymbol{A} + \boldsymbol{B})\boldsymbol{u} = \boldsymbol{A}\boldsymbol{u} + \boldsymbol{B}\boldsymbol{u}$, per ogni $\boldsymbol{u} \in \mathcal{V}$
- Prodotto:$\boldsymbol{A} \in Lin$, $\lambda \in \mathbb{R}$, $(\lambda, \boldsymbol{A}) \mapsto \lambda\boldsymbol{A} \in Lin$, t.c. $(\lambda\boldsymbol{A})\boldsymbol{u} = \lambda(\boldsymbol{A}\boldsymbol{u})$, per ogni $\boldsymbol{u} \in \mathcal{V}$
- Oss. *Lin* con le operazioni introdotte è uno spazio vettoriale.
- Prodotto di composizione: $\boldsymbol{A}, \boldsymbol{B} \in Lin \mapsto \boldsymbol{A}\boldsymbol{B} \in Lin$, t.c. $(\boldsymbol{A}\boldsymbol{B})\boldsymbol{u} = \boldsymbol{A}(\boldsymbol{B}\boldsymbol{u})$, per ogni $\boldsymbol{u} \in \mathcal{V}$. Oss. In generale $\boldsymbol{A}\boldsymbol{B} \neq \boldsymbol{B}\boldsymbol{A}$.

Applicazioni lineari: diade

- $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{V}$, $\boldsymbol{a} \otimes \boldsymbol{b} \in Lin$, $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{u} = (\boldsymbol{b} \cdot \boldsymbol{u})\boldsymbol{a}$, per ogni $\boldsymbol{u} \in \mathcal{V}$ (prodotto diadico o tensoriale)
- Oss. $(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{u} \;//\; \boldsymbol{a}$
- $\boldsymbol{e}$ versore, $(\boldsymbol{e} \otimes \boldsymbol{e})\boldsymbol{u} = (\boldsymbol{u} \cdot \boldsymbol{e})\boldsymbol{e}$ (proiettore)



- $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$, terna cartesiana, $\boldsymbol{I} = \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$
- $\boldsymbol{e}$ versore, $(\boldsymbol{I} - \boldsymbol{e} \otimes \boldsymbol{e})\boldsymbol{u} = \boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{e})\boldsymbol{e}$ (proiettore ortogonale)



Rappresentazione cartesiana delle applicazioni lineari

- $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$...terna destra di versori mutuamente ortogonali
- $\boldsymbol{A} \in Lin(\mathcal{V})$, $A_{ij} = \boldsymbol{e}_i \cdot \boldsymbol{A}\boldsymbol{e}_j$
- $$[\boldsymbol{A}] = \begin{array}{c} \left[\begin{array}{ccc} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{array} \right] \\ \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \boldsymbol{A}\boldsymbol{e}_1 & \boldsymbol{A}\boldsymbol{e}_2 & \boldsymbol{A}\boldsymbol{e}_3 \end{array} \end{array}$$
- $\boldsymbol{v} = \boldsymbol{A}\boldsymbol{u}$ in componenti dà $v_i = \sum_{j=1}^{3} A_{ij} u_j$

$$\left[\begin{array}{c} v_1 \\ v_2 \\ v_3 \end{array} \right] = \left[\begin{array}{ccc} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{array} \right] \left[\begin{array}{c} u_1 \\ u_2 \\ u_3 \end{array} \right]$$

- $(\boldsymbol{a} \otimes \boldsymbol{b})_{ij} = a_i b_j$

Trasposta

- Data $\boldsymbol{A} \in Lin(\mathcal{V})$ esiste unica $\boldsymbol{A}^T \in Lin(\mathcal{V})$ tale che $\boldsymbol{Au} \cdot \boldsymbol{v} = \boldsymbol{A}^T \boldsymbol{v} \cdot \boldsymbol{u}$ per ogni $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V}$

Proprietà

1. $(\boldsymbol{A} + \boldsymbol{B})^T = \boldsymbol{A}^T + \boldsymbol{B}^T$
2. $(\lambda \boldsymbol{A})^T = \lambda \boldsymbol{A}^T$
3. $(\boldsymbol{AB})^T = \boldsymbol{B}^T \boldsymbol{A}^T$
4. $(\boldsymbol{a} \otimes \boldsymbol{b})^T = \boldsymbol{b} \otimes \boldsymbol{a}$
5. $[\boldsymbol{A}^T] = [\boldsymbol{A}]^T$

Applicazioni simmetriche e antisimmetriche

- $\boldsymbol{S}$ simmetrica sse $\boldsymbol{S} = \boldsymbol{S}^T$ $(\boldsymbol{S} \in Sym)$
- $[\boldsymbol{S}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}$ $(S_{ij} = S_{ji})$

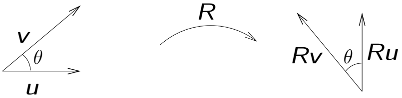
- $\boldsymbol{W}$ antisimmetrica sse $\boldsymbol{W} = -\boldsymbol{W}^T$ $(\boldsymbol{W} \in Skw)$
- $[\boldsymbol{W}] = \begin{bmatrix} 0 & W_{12} & W_{13} \\ -W_{12} & 0 & W_{23} \\ -W_{13} & -W_{23} & 0 \end{bmatrix}$ $(W_{ij} = -W_{ji})$

- $\boldsymbol{w} = (-W_{23}, W_{13}, -W_{12})$ ... vettore assiale, t.c. $\boldsymbol{Wv} = \boldsymbol{w} \times \boldsymbol{v}$

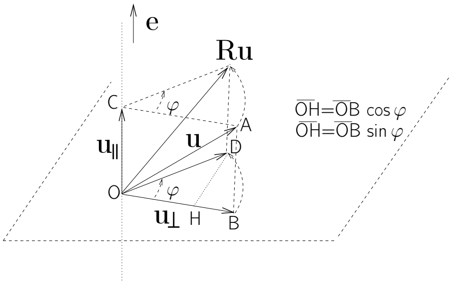
- $\boldsymbol{A} = sym\boldsymbol{A} + skw\boldsymbol{A}$, $sym\boldsymbol{A} = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T)$, $skw\boldsymbol{A} = \frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^T)$

Rotazioni

- $\boldsymbol{e} \in \mathcal{V}$ ... versore
- $\varphi \in [0, 2\pi[$
- $\boldsymbol{W} \in Skw$ tale che $\boldsymbol{Wu} = \boldsymbol{e} \times \boldsymbol{u}$ per ogni $\boldsymbol{u} \in \mathcal{V}$
- $\boldsymbol{R} = \boldsymbol{e} \otimes \boldsymbol{e} + \cos\varphi(\boldsymbol{I} - \boldsymbol{e} \otimes \boldsymbol{e}) + \sin\varphi \boldsymbol{W}$ (formula di rappr. di Eulero)
- $\boldsymbol{Re} = \boldsymbol{e}$ $\qquad\qquad$ $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$ (Dim. vedi dopo)
- $\boldsymbol{Ru} \cdot \boldsymbol{Rv} = \boldsymbol{u} \cdot \boldsymbol{v}$ per ogni $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{V} \;\Rightarrow\; |\boldsymbol{Ru}| = |\boldsymbol{u}|$ per ogni $\boldsymbol{u} \in \mathcal{V}$
- $|\boldsymbol{Ru}|\,|\boldsymbol{Rv}| \cos\widehat{(\boldsymbol{Ru})(\boldsymbol{Rv})} = |\boldsymbol{u}|\,|\boldsymbol{v}| \cos\widehat{\boldsymbol{uv}} \;\Rightarrow\; cos\widehat{(\boldsymbol{Ru})(\boldsymbol{Rv})} = \cos\widehat{\boldsymbol{uv}}$



Rotazioni



- $\boldsymbol{Ru} = \boldsymbol{Ru}_{||} + \boldsymbol{Ru}_{\perp}$
- $\boldsymbol{Ru}_{||} = \boldsymbol{u}_{||} = C - O$
- $\boldsymbol{Ru}_{\perp} = \cos\varphi\, \boldsymbol{u}_{\perp} + \sin\varphi\, \boldsymbol{e} \times \boldsymbol{u}_{\perp} = D - O$
- $\boldsymbol{R}$ ... rotazione di ampiezza φ e asse $\boldsymbol{e}$
- Oss. Al variare di $\boldsymbol{e}$ e φ si generano tutte le rotazioni dello spazio

Rotazioni (rappresentazione cartesiana)

- $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$...terna cartesiana
- $\boldsymbol{e}_3 \equiv \boldsymbol{e}$
- $\boldsymbol{R}\boldsymbol{e}_1 = \cos\varphi \boldsymbol{e}_1 + \sin\varphi \boldsymbol{e}_2$
- $\boldsymbol{R}\boldsymbol{e}_2 = -\sin\varphi \boldsymbol{e}_1 + \cos\varphi \boldsymbol{e}_2$
- $\boldsymbol{R}\boldsymbol{e}_3 = \boldsymbol{e}_3$
- $[\boldsymbol{R}] = \begin{bmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (Verificare $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$)

Rotazioni infinitesime

- $\boldsymbol{R} = \boldsymbol{I} + \varphi \boldsymbol{W} + o(\varphi) = \boldsymbol{I} + \bar{\boldsymbol{W}} + o(\varphi)$, $\bar{\boldsymbol{W}} = \varphi \boldsymbol{W}$ per $\varphi \to 0$
- $\boldsymbol{R}\boldsymbol{u} = \boldsymbol{u} + \varphi \boldsymbol{W}\boldsymbol{u} = \boldsymbol{u} + \varphi \boldsymbol{e} \times \boldsymbol{u} = \boldsymbol{u} + \boldsymbol{w} \times \boldsymbol{u}$
- $\boldsymbol{w} = \varphi \boldsymbol{e}$ ... vettore rotazione di $\bar{\boldsymbol{W}}$